CARPE DONUT

How much should people pay for donuts?

lesson guide

Carpe Donut in Charlottesville, Virginia, has an interesting pricing scheme. You can buy one donut for $2, two donuts for $3, three for…well, you get the idea. This means that two people could pay less by purchasing their donuts together. Three people could do even *better*. So how does the average cost per donut change, and how much should we be paying for each?

In this lesson, students use linear, rational, and piecewise functions to describe the total and average cost of an order at Carpe Donut. Through this, they'll see the benefits of buying in bulk, and will find a least-expensive way to purchase these delicious donuts.

Primary Objectives

- Use linear and piecewise-linear functions to describe a pricing scheme
- Model average cost using rational functions
- Make formal or informal limit arguments to explain end behavior of total and average cost functions

Content Standards (CCSS)	Mathematical Practices (CCMP)	Materials
Functions: IF.5, IF.6, IF.7.b, IF.7.d, BF.1.a, BF.1.b	MP.3, MP.4, MP.7	• Student handout • LCD projector • Computer speakers

Before Beginning…

Students should be able to graph and write equations for linear functions. They should also know how to calculate an average. Some previous exposure to piecewise-defined functions (the floor function in particular) will also be helpful. Limits make an appearance, but prior experience with them is not necessary. Lastly, although rational functions play an important role in the lesson, we assume no previous experience.

Preview & Guiding Questions

Students view a photograph of Carpe Donut in Charlottesville, Virginia, and learn that a single donut will cost you $2. Discounts quickly kick in, though: two donuts only cost $3, three cost $4, and so on. In groups, have students strategize how they would order if they went to this donut stand together. Should each person order his or her own donut, or should one person order for the group? Or should they do something in between?

The goal is for students to realize that, no matter what, they'll save the most if one person orders for everyone. In a group of four, for example, if one person orders four donuts, the total will only be $5, compared to $8 if everyone orders their own donut. If students have trouble with this realization, you can ask more specific questions.

After informally exploring these prices, students will be ready to dive in to the math behind Carpe Donut's pricing.

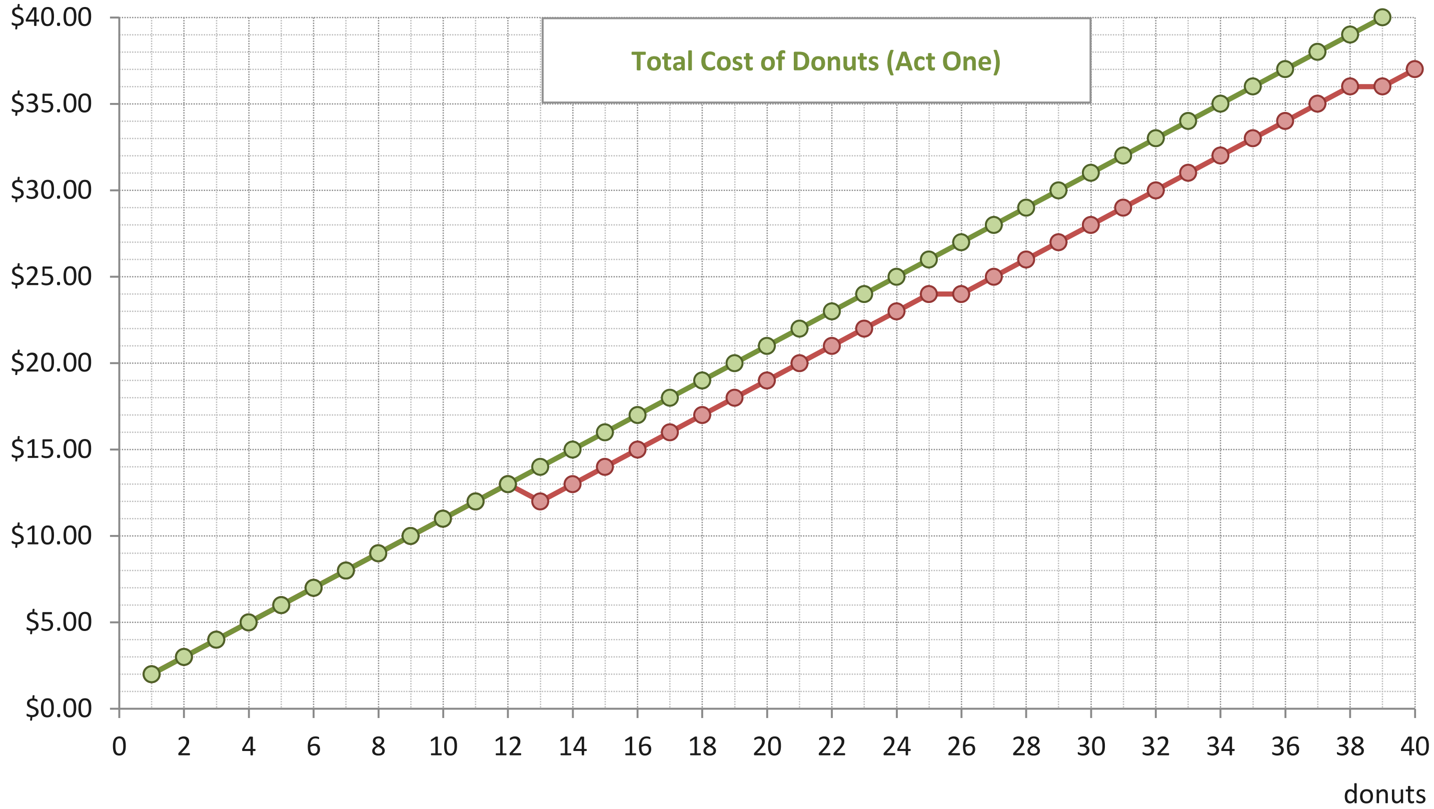
- *What would the total be if one of you ordered all the donuts?*
- *If using groups of four: What would the total cost be if two of you each ordered two donuts?*
- *What would the total be if each of you ordered your own donut?*
- *How would your group order if you wanted to save as much money as possible?*

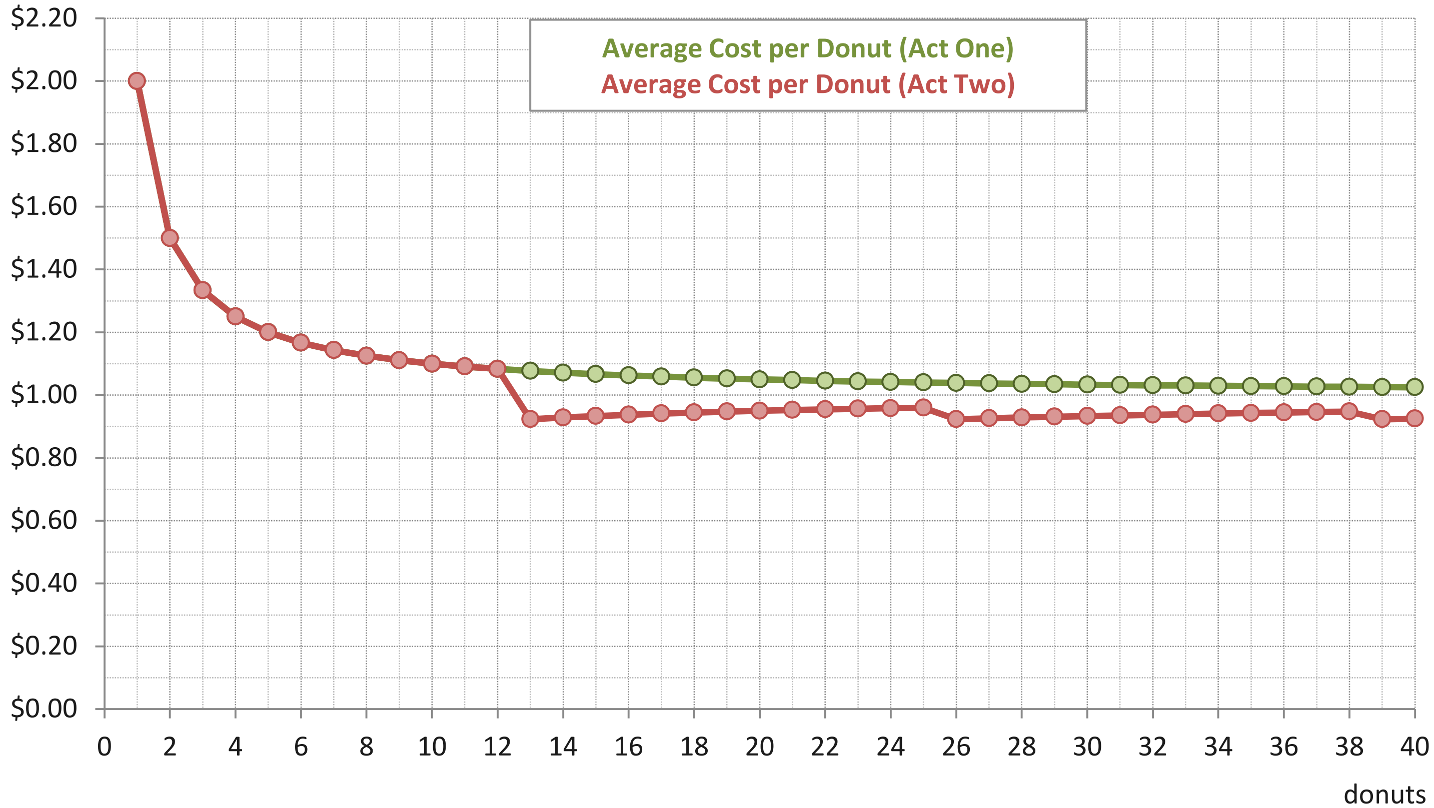
Act One

In Act One, students will develop a mathematical model for Carpe Donut's pricing. Using this model, they'll discuss how much they would charge a person behind them in line who asked for a donut. Then, they will create a model for the average cost when someone buys ***d*** donuts. Finally, by analyzing the behavior of the average cost function, they'll discover that as you buy more donuts, the cost of each one gets closer and closer to $1.

Act Two

If you thought the pricing at Carpe Donuts was already weird, in Act Two we discover that things get even stranger. This is due to the fact that they give you price breaks after every 13 donuts you buy. For example, a dozen donuts costs $13, but thirteen donuts only costs $12! Students will modify their model to reflect this new pricing structure, and will come up with both a piecewise-linear cost function and a really funky looking average cost function. We then turn to the same question post in Act One: what happens to the average cost as you buy more and more donuts? The answer may surprise you (and your students)!


Total Cost of Donuts (Act One)
$40.00
$35.00
$30.00
$25.00
$20.00
$15.00
$10.00
$5.00
$0.00
0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40
donuts


Average Cost per Donut (Act One)
Average Cost per Donut (Act Two)
$2.20
$2.00
$1.80
$1.60
$1.40
$1.20
$1.00
$0.80
$0.60
$0.40
$0.20
$0.00
0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40
donuts

Act One: Donut Stand

1 There's a donut stand in Charlottesville, Virginia, that sells homemade donuts. It charges \$2 for one donut, \$3 for two donuts, \$4 for three donuts, etc. Write a function to calculate the total cost of ***d*** donuts and graph it.

They charge \$1 per donut, plus an extra dollar for the first one, which means their prices can be modeled by

$$T(d) = d + 1, \text{ for } d \geq 1,$$

where d is the number of donuts and T is the total cost.

Explanation & Guiding Questions

\$2 for 1, \$3 for 2, \$4 for 3…the pattern here is pretty easy to spot, at least qualitatively. The total cost for donuts is always one more than the number of donuts. After seeing a few examples, students may be able to describe the rule: the cost of d donuts should be $d + 1$ dollars. Or, using function notation, $T(d) = d + 1$. If they need additional scaffolding, encourage them restate the relationship in their own words, for example, "The total cost is always one more than the number of donuts."

Before students start to graph, they should be able to identify T (the total cost) as a linear function of d (the number of donuts). You can then talk about some properties of the function in terms of the context. For example, a constant slope means the cost per additional donut is always the same. The value of the slope tells us that the cost per additional donut is always \$1.

What about the intercept? This tells us that the price for zero donuts is…\$1. Wait, what? Charging people who don't order anything doesn't seem like the savviest (or most enforceable) business move. The problem here is that the domain of our function only makes sense starting at one donut. More refined graphs will better reflect reality.

- *In your own words, how is the total cost of the donuts related to the number of donuts?*
- *If you want d donuts, how much will you have to pay?*
- *What type of function is this? What will the graph look like?*
- *What's the slope of your line, and what does it represent?*
- *What's the y-intercept of your line, and what does it represent?*
- *When does your function begin to make sense? How does this impact the graph?*

Deeper Understanding

- *What's a reasonable domain for your function? (Whole numbers. The pricing doesn't make sense for zero donuts, and it also doesn't seem likely that you could buy fractions of a donut (e.g. 2.5 donuts for \$3.50).)*

2 Imagine you're standing in line, and you're about to buy a single donut. Someone comes up to you and says, "I'll give you $1 if you buy me one, too." Would you accept the deal? If not, how much would you charge and why?

If the person bought the donut on his own, he'd have to spend $2. Two donuts cost $3. This means that the visitor should be willing to spend anywhere from $1 (in which case we'd pay $2) to $2 (in which case we'd pay $1). A reasonable solution might be for each person to pay $1.50.

Explanation & Guiding Questions

At first glance, this may seem like a good idea. The person behind you gets a donut at half price ($1 instead of $2), and you don't have to pay any extra for your own donut.

But wait. You're paying $2 either way, but if you buy a donut for the person behind you, he's only paying for $1 of the $3 total! Why should you be paying twice as much, and not benefitting at all from the discounted total price? A fairer approach would be to split the total cost down the middle, with each of you paying $1.50. That way you each pay less than what you otherwise would have, and the discount is distributed equally between you.

- *How much will it cost if you just order your own donut? How much will it cost if you order both donuts?*
- *If you take the deal, how much will you be paying for your own donut? If you don't take the deal?*
- *If you take the deal, how much will the visitor be paying for his/her donut? If you don't take the deal?*
- *If you take the deal, how much is each of you contributing towards the total cost?*
- *Does this deal seem fair? Why or why not?*
- *What do you think is a fairer amount to pay?*

Deeper Understanding

- *How is the fairest price related to the total donut cost? (It's the average cost, i.e. the cost per donut.)*
- *What's the* ***most*** *you'd consider charging this person, and why? (The person behind you should be willing to pay up to $1.99, since that's still cheaper than buying from the stand. However, he or she may view this as an unfair deal, and might not take it even if it saves some money.)*

3 If everyone in a group of 10 friends wants a donut, how much should each person pay? 20 friends? ***n*** friends?

number of friends	*cost of donuts*	*each person pays*
10	*\$11*	$\frac{\$11}{10} = \$1.10$
20	*\$21*	$\frac{\$21}{20} = \$1.05$
n	*n + 1*	$\frac{n+1}{n} = 1 + \frac{1}{n}$

Explanation & Guiding Questions

In the last question, students saw how they could save money by ordering as a pair. Here we take this idea to the next level by considering larger and larger groups of people.

To figure out how much each person should pay, we first need to know the total cost in each case (e.g. for 10 friends, the price must be $T(10)$ = \$11). There are a number of ways the group could split the total, but it seems like the fairest way is for everyone to pay the same amount. In other words, we should divide the total cost by the number of people (for 10 friends, this results in a cost of \$11/10 = \$1.10 per person). Or, to put it another way, each person should pay the **average cost** of a donut.

Doing this for the first two groups (10 friends and 20 friends) should not pose much of a problem. When it comes to the general case, students repeat the same reasoning with expressions instead of explicit numbers. Breaking down each component of the expression will help them generalize their thinking.

- *With 10 friends, what's the total cost of the donuts? With 20 friends?*
- *What's the fairest way to split the cost?*
- *In each case, how is the fairest way to split the cost related to the average cost of a donut?*
- *How much will each of 10 friends pay? Each of 20 friends?*
- *With n friends, what's the total cost? Into how many pieces will that cost be divided?*
- *What's an expression for how much each person should pay?*

Deeper Understanding

- *How many friends would you need for each of them to pay less than \$1.04? (We'd need $1 + 1/n \le 1.04$, so that $1/n \le 0.04$, or $n \ge 25$.)*
- *What would be the total cost if each friend bought a donut individually, and how much do they save by buying as a group? (The total cost would be 2n dollars, meaning that the group saves $2n - (n + 1) = n - 1$ dollars by ordering together.)*
- *What type of function is the amount each person pays? (A rational function of the number of people, n.)*

4 As you buy more and more donuts, what happens to the average cost of each donut, and how many donuts would you need to buy for the average cost to be $1?

As the number of donuts gets larger, the average cost gets closer to $1 per donut. The average will never quite get to $1 per donut, though, since 1 + 1/n never quite gets to 1. After 200 donuts, however, the average cost dips below $1.005, so it would round down to $1 per donut.

Explanation & Guiding Questions

Students have already seen that the average cost of a donut decreases when ordering 20 donuts instead of 10 ($1.05 vs. $1.10). But does the average cost continue to decrease as you order more donuts? This may be hard for students to answer by simply looking at the expression they wrote down in the previous question.

You might suggest that students graph the average cost function using technology. Based on the graph, the average cost is decreasing as we order more donuts, but not at a constant rate. The graph appears to be getting closer and closer to $1. This makes sense if we think about the average cost equation, which tells us that the average cost of n donuts is equal to

$$\frac{n+1}{n} = 1 + \frac{1}{n}.$$

One way to think of this expression is that the customer is paying a total price of $1 per donut, plus an additional dollar. As the number of donuts gets larger and larger, that additional dollar is getting divided among more and more donuts in the average cost, so that "extra" cost gets smaller and smaller.

Will the average reach a dollar? Mathematically, this would mean that $1/n$ reaches 0, which doesn't happen since no matter how large n is, $1/n$ will still be some positive number. However, practically speaking, the average cost will dip below $1.005 if you order more than 200 donuts. In other words, if you round to the nearest cent, the average cost would hit a dollar for orders that are large enough.

- *What happened to the average cost when we went from 10 to 20 donuts?*
- *Graph the average cost function. What happens to the function as you order more donuts?*
- *What's another way to write $\frac{n+1}{n}$, and what does this tell you about the average price of a donut?*
- *Theoretically, will the price of a donut ever reach $1? Explain.*
- *Practically, will the price of a donut ever reach $1? Explain.*

Deeper Understanding

- *Will the average cost ever be below $1? (No, since 1 + 1/n is always greater than 1 when n is positive.)*
- *Will the average cost ever be above $2? (No, since 1 + 1/n is always at most 2 when n is at least 1.)*
- *Will the average cost ever equal $1.15? (No, since this would imply that 1/n = 0.15, or n ≈ 6.7. But if we round to the nearest cent, the average cost will be $1.17 when n = 6, and $1.14 when n = 7.)*
- *What's a reasonable domain for the average cost function? (It still doesn't make sense to talk about fractions of donuts, so the domain should be the set of whole numbers.)*

Act Two: Baker's Dozen

5 Like many bakeries, Carpe Donut offers a special "baker's dozen:" if you buy thirteen donuts, they'll only charge you \$12. After that, additional donuts cost \$1, but every thirteenth donut is free. Based on this, how much will you pay for the following numbers of donuts?

1	2	3	4	5	6	7	8	9	10
\$2	\$3	*\$4*	*\$5*	*\$6*	*\$7*	*\$8*	*\$9*	*\$10*	*\$11*
11	12	13	14	15	16	17	18	19	20
\$12	\$13	\$12	\$13	*\$14*	*\$15*	*\$16*	*\$17*	*\$18*	*\$19*
21	22	23	24	25	26	27	28	29	30
\$20	*\$21*	*\$22*	*\$23*	*\$24*	*\$24*	*\$25*	*\$26*	*\$27*	*\$28*
31	32	33	34	35	36	37	38	39	40
\$29	*\$30*	*\$31*	*\$32*	*\$33*	*\$34*	*\$35*	*\$36*	*\$36*	*\$37*

Explanation & Guiding Questions

If only life were as simple as the function $T(d) = d + 1$ suggests. In reality, though, the pricing at Carpe Donut is a bit complicated. The purpose of Act Two is to understand this added complexity, and describe it mathematically.

First, students should understand that Carpe Donut's generosity means that their function $T(d)$ from Act One no longer works. While $T(12)$ = \$13, as expected, $T(13)$ = \$12. Additional donuts then cost a dollar each, as expected, until we hit 26, since both 25 and 26 donuts cost \$24. After this small hiccup, the total cost continues to rise at a rate of \$1 per donut, until we hit 39 donuts. This time, both 38 and 39 donuts cost \$36.

Students may notice that the pricing gets a little weird around every baker's dozen. Since every thirteenth donut is free after the first baker's dozen, the pricing will temporarily halt its ascent around multiples of thirteen. Noticing this pattern will help students come up with a modified price function in the next question.

- *Is T(d) = d + 1 still a valid model? Why not?*
- *What's the total cost of twelve donuts? Thirteen? Fourteen?*
- *After thirteen donuts, when's the next time the total cost doesn't increase by \$1?*
- *What's the total cost of 25 donuts? 26? 27?*
- *After 26 donuts, when's the next time the total cost doesn't increase by \$1?*
- *Do you notice any pattern emerging? Can you describe it?*

Deeper Understanding

- *How many times does total cost* ***decrease*** *when you increase the number of donuts? (Only once, when moving from twelve to thirteen donuts).*
- *How many times does total cost* ***stay the same*** *when you increase the number of donuts (Infinitely many times.)*

6 Given the new "baker's dozen" scheme, write a function to calculate the total cost of *d* donuts and graph it.

Up through 12 donuts, the total cost is just d + 1, as before. From 13 through 25 donuts, the total cost is d – 1. Starting with the next baker's dozen (26 donuts), the total cost is d – 2. In other words, after the first baker's dozen, the total cost of d donuts equals

d – (# of baker's dozens).

The number of baker's dozens is equal to the quotient d and 13, ignoring the remainder. This can be expressed using the floor (or int) function. So the final total cost function looks like:

$$T(d) = \begin{cases} d + 1, & 1 \leq d < 13, \\ d - \left\lfloor \frac{d}{13} \right\rfloor, & d \geq 13. \end{cases}$$

Explanation & Guiding Questions

After filling out the table, there should be no doubt in students' minds that their total cost function from Act One is no longer a perfect model. But before throwing out all of their hard work, it's worth pointing out that the old model *does* work within a limited domain. Namely, $T(d) = d + 1$ provided $d < 13$.

What happens for larger numbers of donuts? To help them visualize the pattern, it may be best for students to graph the data in their table before they try to come up with an equation. Once they've graphed this new function, they may notice that between 13 and 25 donuts, the new total cost is just the old price shifted down. From 26 to 38 donuts, the total cost shifts down a bit more.

This is also reflected in the table. From 13 to 25 donuts, the total cost is $d - 1$, and from 26 to 38 the total cost is $d - 2$. In other words, after each subsequent baker's dozen, we're subtracting another dollar from the total cost.

If students can verbalize this pricing rule when $d > 13$, that's great. If you want them to write things down more precisely, you'll need to use the floor function $\lfloor x \rfloor$. This function rounds any number down to the nearest whole number: $\lfloor 2.1 \rfloor = 2$, $\lfloor 4.99 \rfloor = 4$, and so on. Since the number of baker's dozens in *d* donuts is the whole number part of $d \div 13$, we can also write this using the floor function as $\lfloor d/13 \rfloor$. But this is exactly the number that's getting subtracted in the total cost function when $d \geq 13$!

- *After how many donuts does your function T(d) = d + 1 stop working?*
- *How does your new price graph differ from the previous one?*
- *Can you come up with a new function that describes the total cost between 13 and 25 donuts?*
- *Can you come up with a new function that describes the total cost between 26 and 38 donuts?*
- *What number are you subtracting in each of these new functions?*
- *What's the value of $\lfloor 2.1 \rfloor$? $\lfloor 4.99 \rfloor$?*
- *How can you describe the number of baker's dozens in d donuts using the floor function?*
- *How can you describe the total cost function for any number of donuts?*

Deeper Understanding

- *Is this new function linear? Why or why not? (No. The price per additional donut is not constant. Also, the graph is no longer a straight line.)*
- *Is it ever true that $\lfloor x \rfloor > x$? (No. Since the floor function always rounds down, $\lfloor x \rfloor \leq x$, and equality only holds if x is an integer.)*

7 As you buy more and more donuts, what happens to the average cost of each one? Be as specific as possible.

The break in price after each baker's dozen keeps the total cost under what it was before the discount, which means the average cost should stay less than \$1. As the number of donuts increases, customers are essentially paying \$12 for every 13 donuts, so the average cost should be getting closer to \$12/13, or about \$0.92 per donut.

More formally: We can rewrite $\left\lfloor \frac{d}{13} \right\rfloor$ *as* $\frac{d}{13} - \left\{\frac{d}{13}\right\}$*, where the curly brackets indicate the fractional part left over after using the floor function. (E.g. if d = 15, then* $\left\lfloor \frac{d}{13} \right\rfloor = 1$ *and* $\left\{\frac{d}{13}\right\} = \frac{2}{15}$*.)*

After making this substitution, we can write the average cost for d $\underline{>}$ *13 as* $\frac{1}{d}\left(d - \frac{d}{13} + \left\{\frac{d}{13}\right\}\right) = \frac{12}{13} + \frac{1}{d}\left\{\frac{d}{13}\right\}$*. Since the fractional part of a number is always between 0 and 1,* $\lim_{d\to\infty}\left(\frac{1}{d}\left\{\frac{d}{13}\right\}\right) = 0$*. Therefore, the limit exists, and*

$$\lim_{d\to\infty}\left(\frac{d - \frac{d}{13} + \left\{\frac{d}{13}\right\}}{d}\right) = \lim_{d\to\infty}\left(\frac{\frac{12}{13}d}{d}\right) + \lim_{d\to\infty}\left(\frac{1}{d}\left\{\frac{d}{13}\right\}\right) = \frac{12}{13}$$

Explanation & Guiding Questions

The average cost function is far from intuitive, but the process is the same as before. In principle, all students must do to find the average cost is divide the total cost by the number of donuts. Understanding this complicated function, however, will prove to be more of a challenge.

It may be best to start by having students write down the average cost for *d* < 13, since this is the same expression they found in Act One. For *d* ≥ 13 the average cost becomes

$$\frac{d - \left\lfloor \frac{d}{13} \right\rfloor}{d} = 1 - \frac{1}{d}\left\lfloor \frac{d}{13} \right\rfloor,$$

which isn't exactly the easiest thing to wrap your head around. From here, have students graph their average cost functions and start looking for patterns. Students should quickly notice that the average cost drops below \$1 and appears to stay there. In fact, the average cost drops after every 13th donut, then rises slightly with each additional donut until the next baker's dozen is reached. If students notice that the average cost is \$12/13 after each baker's dozen, they may guess that this is the new average value. And they'd be right! Proving this, though, may be too difficult – if you're feeling bold enough, you can try to walk students through the limit argument provided above.

- *What's the average cost of d donuts when d < 13? When d ≥ 13?*
- *What does the graph of average cost look like now?*
- *In general, does the average cost appear to be higher than, lower than, or equal to \$1?*
- *Do you think the average cost is heading towards a value as the number of donuts increases? If so, what value might it be? If not, why not?*
- *What's the average cost of 13 donuts? 26 donuts? 39 donuts?*
- *Can you describe what happens to the average cost when d gets large just by looking at the equation?*
- *What happens to* $\frac{12}{13} + \frac{1}{d}\left\{\frac{d}{13}\right\}$ *as d gets large?*

Deeper Understanding

- *Once the average cost drops below \$1, will it ever rise above \$1 again? (No. This is because the average price first drops below \$1 when d = 13, and for d > 13,* $\frac{12}{13} + \frac{1}{d}\left\{\frac{d}{13}\right\} \leq \frac{12}{13} + \frac{1}{d} < \frac{12}{13} + \frac{1}{13} = 1$*.)*
- *Will the average cost ever drop below \$12/13? (No, since* $\frac{1}{d}\left\{\frac{d}{13}\right\} \geq 0$*.)*