GOING ONCE, GOING TWICE

lesson guide

Driving Question	Total Time
How much should you bid at an auction?	90 minutes

Real-World Takeaways

- In an auction, a higher bid has a higher probability of winning (than a lower one), but reduces your profit should you win. An optimal bid can be thought of as the one with the highest expected value.
- In a two person auction, the optimal bid is approximately half of the item's value.
- As more people participate in an auction, the probability of any given bid winning decreases (because it has to beat multiple people), so the optimal bid increases.

Math Objectives and Common Core Standards

• Write and graph a quadratic equation to model a real-world situation. • Write and graph higher-order polynomial equations to describe a real-world situation. • Calculate and model expected value in a real-world context	High School F.BF.A.1 A.CED.A.3 S.MD.B.5

Question Overview and Suggested Time

Preview			5 minutes
Act One	1	Calculate the profit gained from bidding different amounts on an item secretly valued at $80. Model the profit as *$80 - b*, where b = bid amount.	5 minutes
	2	Calculate the probability of outbidding one other person who also values the item at $80, given various bid amounts. Model the probability as b/$80, where b = bid amount.	15 minutes
	3	Calculate the expected value of various bids. Write and graph an equation to describe expected value in terms of b: *($80-b)(b/$80),* using the previously generated expressions for profit and probability. Identify the optimal bid at the parabola's maxima.	20 minutes
Act Two	4	Describe how adding a third bidder would affect profit, probability, and expected value of a bid. Revise the equation from Q3 to model three bidders, sketch the graph, and find the new expected-value-maximizing bid.	25 minutes
	5	Describe how additional bidders affect the optimal bid.	20 minutes

LESSON PREVIEW/HOOK

Students watch a clip from the NBC comedy, *The Office*. The staff is holding a silent auction, but Dwight confuses it for a game of guess-the-price. “You think jiu jitsu classes cost $22?” he says. “If you’re going to guess the price, you might as well be halfway accurate.” He then takes the clipboard and writes “$180.” Needless to say, Dwight ends up winning lots of items and spending lots of money. If only he realized that the goal of an auction isn’t just to win the item; it’s to win it for as little money as possible!

There are lots of different types of auctions. Sometimes bids are public, and sometimes they aren’t. Sometimes people can bid multiple times, and sometimes only once. Ask students to imagine that you’re holding **a sealed bid auction** for [an item that students would want, e.g. Bluetooth headphones or a gift card to their favorite store]. A sealed bid auction is one in which each participant can submit only one bid, and their bid is not known to others in the auction. Once the class agrees on a value -- e.g. *the headphones cost $80 on Amazon* -- ask them to secretly write a bid amount. When they’re done, have students share their bids and discuss how they decided on them.

Discussion Questions

1. What’s happening in the video clip? Does it appear that Dwight understands how an auction works?
2. What is a silent auction, and how is it different than other types of auctions you might have seen?
3. What are some of your favorite products and/or places to shop?
4. Imagine we’re holding a sealed bid auction for [product]. Let’s assume the item is worth [value]. Write down a secret bid.
5. Who would have won the auction, and how did you decide how much to bid?

Key Takeaway

- The goal of an auction is not just to win the item, but to win it for as little as possible. This means that participants often strategize when deciding how much to bid, e.g. by guessing the maximum bid their opponents might offer, then bidding slightly more than this.

1 Imagine you're participating in a **sealed-bid auction** in which you submit a single, secret bid. You value the item at $80, but you'd like to spend as little as possible to get it; if you can win with a bid of $8, for instance, you'll effectively make a "**profit**" of $72. With each bid below, how much profit would you make if you won?

Bid	$8	$20	$32	$48	$64	$80
Profit	$72	$60	$48	$32	$16	$0

b
80 – b

Guiding Questions

1. If you win an $80 item with a bid of $20, how much profit would you make? What if you win with a bid of $80?
2. If you win an $80 item with a bid of *b* dollars, how can you quickly determine the profit?

Checks for Understanding

- What does it mean to "profit" in an auction? *(Relative to flat-out buying the item for $80, the profit describes how much better off you are by getting it for less.)*
- Would it be possible for the profit to be negative, i.e. to take a loss? *(Yes, if you win with a bid of more than $80.)*

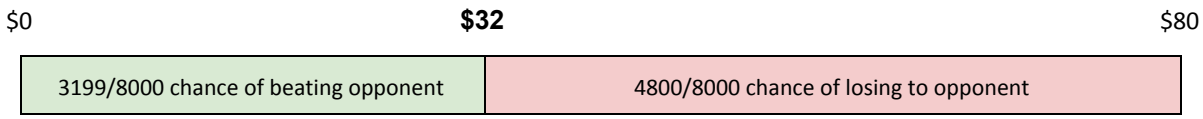
2 For simplicity, let's assume you're only competing against one other person in the auction; since you know nothing about how much your opponent values the item, let's also assume (i) he/she values it exactly the same as you do, and (ii) he/she is equally likely to place any bid from \$0.01 to \$80.00. Based on this, what is your **probability of winning** the auction with each bid amount below?

Bid	\$8	\$20	\$32	\$48	\$64	\$80
P(win)	10%	25%	40%	60%	80%	100%
	$\frac{8}{80}$	$\frac{20}{80}$	$\frac{32}{80}$	$\frac{48}{80}$	$\frac{64}{80}$	$\frac{80}{80}$

b
b / 80

Note to Teacher

Students may reason that an \$80 bid won't necessarily win the auction; after all, your opponent could also bid \$80, in which case nobody would win. Similarly, they may reason that a \$32 bid doesn't have a 40% chance of winning but slightly less. This is true. Since the problem indicates that the auction allows bids in pennies, there are 8000 possible bids in total. Since a \$32.00 bid would beat 3199 of these, the probability of winning with it is 3199/8000 = 39.9875%, not 40%. Likewise, the probability of winning with an \$80 is 7999/8000 = 99.9875%, not 100%. Thus, a more precise expression for the probability of winning with a bid of b is ***(b – 0.01) / 80****, which could also be written as* ***(100b – 0.01) / 8000****.*



However, since the likelihood of tying in a 2-person auction is only 1 in 8000, we think it's reasonable to ignore this edge case and approximate the probability of winning as ***b / 80****. Here, P(win) is 40% with a bid of \$32 and 100% with a bid of \$80.*

Guiding Questions

1. If you bid \$32, what would need to happen for you to win the auction?
2. If you bid \$32, what's the likelihood that your opponent bids less than \$32?
3. In an auction that allows pennies, how many possible bids are there between \$0 and \$80?
4. Of these, how many would a \$32 bid beat? Based on this, what's the exact probability of winning with a \$32 bid?
5. In a 2-person auction, what's the probability of tying? Given how unlikely this is, is it reasonable to ignore it?
6. If we ignore the possibility of a tie, what's the probability of winning with a bid of \$32? With a bid of b dollars?

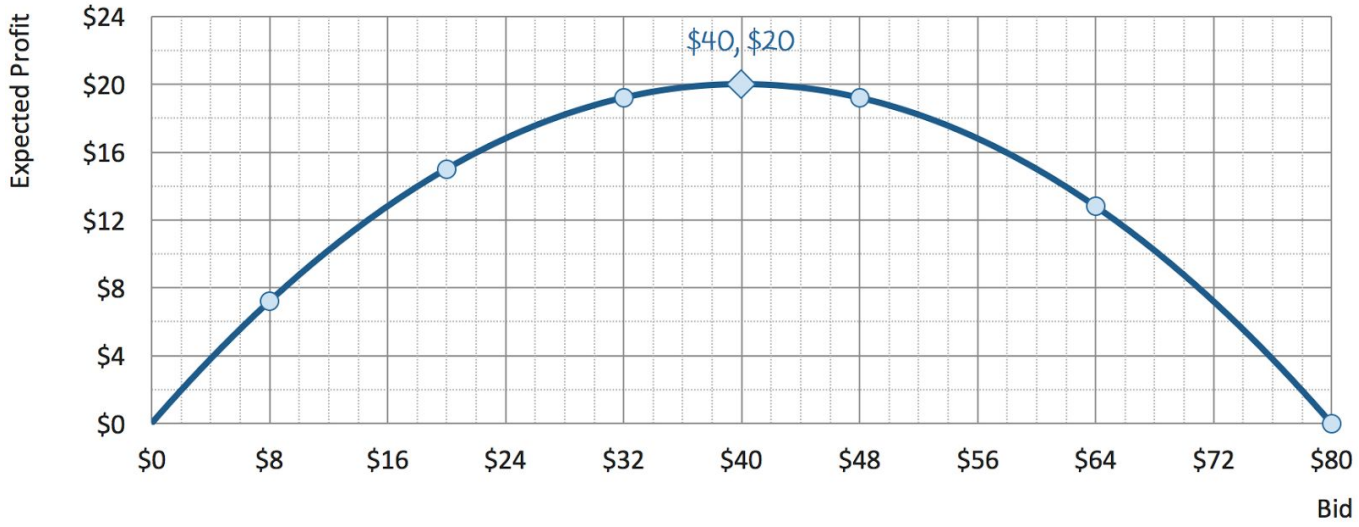
Checks for Understanding

- Is it true that an \$80 bid has a 100% chance of winning? *(No. It has a 7999/8000 = 99.9875% chance of winning.)*
- Since the auction allows bids in pennies, what's a more precise expression for P(win)? *((b – 0.01) / 80)*
- If the auction only allowed bids in dollars, what would the precise expression be? *((b – 1) / 80)*
- Why is it important that we assume the opponent values the item at \$80 and also that he/she bids at random? *(If the opponent valued the item at something other than \$80, and if his/her bids were not random, then we'd have no way to determine the probability of winning associated with a given bid. Even though these assumptions are unrealistic -- everyone values things differently, and people are probably more likely to bid \$20 than \$0.01 -- since we have no information about the opponent, the assumptions are still okay to make.)*

3 If a certain bid offers a 25% chance of yielding a \$60 profit, we say it has an **expected profit** of 0.25 × \$60 = \$15. Write an equation for the relationship between the bid amount, *b*, and the expected profit, and sketch it below. If you value the item at \$80, how much do you think you should bid and why?

Bid	\$8	\$20	\$32	\$48	\$64	\$80
Expected Profit	\$7.20	\$15.00	\$19.20	\$19.20	\$12.80	\$0

b
$(80 - b) \times (b / 80) = (80b - b^2) / 80$



If I bid \$80, I'd win the item but I wouldn't really come out ahead; after all, the item is worth exactly \$80 to me; here, the expected profit is \$0. On the other hand, if I were able to win the item for \$8, I'd basically \$72. However, the probability of this happening is low, and my expected profit with an \$8 bid is only \$7.20. The expected profit increases as I increase the bid amount, until it reaches a maximum and begins to come down again. Since my goal isn't to maximize my of winning but rather to maximize how much better off I'd be , I think the optimal bid is \$40. This is where the expected profit graph reaches its highest point of \$20.

Guiding Questions

1. For an \$80 bid, what is the corresponding profit and probability of winning? For an \$8 bid?
2. What is the expected profits associated with a bid of \$80? With a bid of \$8?
3. For a given bid, b, how can you write an expression for the expected profit?
4. What shape does the graph of (80 – b) × (b / 80) make? What does it tell you about the relationship between the bid amount and the expected profit?
5. How can you use the graph to determine your optimal bid?

Checks for Understanding

- What type of function is the expected profit and why? *(It's quadratic. The expected profit is found by multiplying the profit associated with a bid times its probability of winning. Since the profit and p(win) are both linear functions, their product -- the expected profit -- will be quadratic.)*

4 Imagine a third person joins the auction. For a given bid, *b*, briefly describe how this second opponent will affect your profit and your probability of winning the auction. Then write an equation for the new expected profit and sketch it below. What is your optimal bid now?

Profit

If I win the auction with a bid of \$20, I'll still make a profit of (\$80 – \$20) = \$60. Adding a second opponent shouldn't affect the profit. Therefore the profit function is the same no matter how many opponents there are.

Profit = 80 – b
where b = bid amount in dollars

P(win)

With a \$20 bid, I have a 25% chance of beating Opponent A and a 25% chance of beating Opponent B. (They're independent events.)

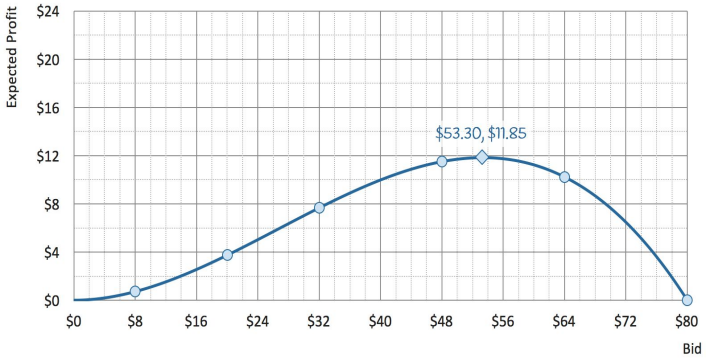
Possible outcomes	Beat B (25%)	Lose to B (75%)
Beat A (25%)	0.25 × 0.25 = 0.0625	0.25 × 0.75 = 0.1875
Lose to A (75%)	0.75 × 0.25 = 0.1875	0.75 × 0.75 = 0.5625

With a bid of \$20, the probability that I'll beat opponents is (0.25) × (0.25) = 0.0625 = 6.25%. For any given bid, b:
Probability of Winning = $(b / 80)^2$

Expected Profit

For a given bid, b, the expected profit is:
Profit × Probability of Winning =
$(80 – b) \times (b / 80)^2 = (80b^2 – b^3) / 80^2$

When I only had one opponent, the optimal bid was \$40 and the corresponding expected profit was \$20. With two opponents, the optimal bid is higher (\$53.30) and the expected profit is lower (\$11.85).



Guiding Questions

1. If a third person joins the auction, how much profit would you make if you won with a bid of \$20?
2. If you bid \$20, what's the probability that you would beat Opponent A? That you would beat Opponent B?
3. If you bid \$20, what's the probability that you would beat both Opponent A *and* Opponent B?
4. What are the different outcomes in a 2-person auction, and what's the probability of each outcome?
5. How can we adjust the expected profit expression for a one-opponent auction to work for two opponents?

Checks for Understanding

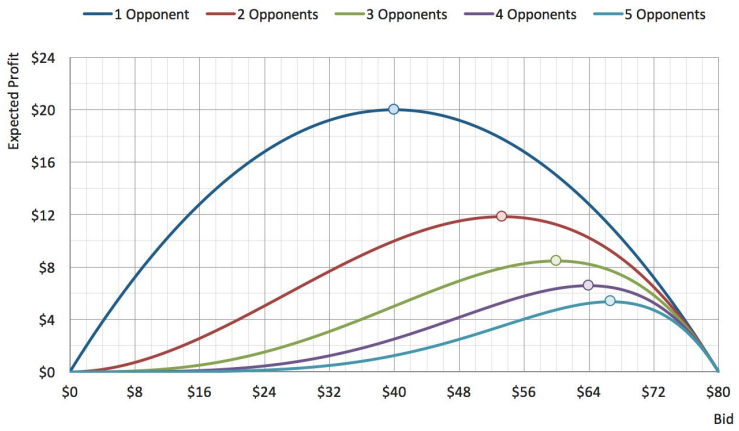
- If a third opponent joined the auction, how would the expected profit function change? *(The exponent would be 3.)*
- As more people join the auction, will the probability of winning always go down? *(If your bid is less than \$80, yes.)*
- What's the relationship between the number of opponents and the power of the function? *(With one opponent, the function is quadratic; it has a power of 2. With two opponents, the function is cubic; it has a power of 3. The power of the function is one more than the number of opponents. Alternatively, we can say the power of the function is equal to the number of **people** in the auction. Applying this to a one-person auction, we get an expected profit of $(80 – b) \times (b / 80)^0 = (80 – b)$. This makes sense: If you have no opponents, your profit will be your bid!)*

5 In reality, auctions can have hundreds – even thousands – of participants. As more people join your auction and compete with you for the $80 item, how would you adjust your bidding strategy? Put another way, as the number of opponents in an auction increases, what happens to the optimal bid and why?

For p opponents, the probability of winning is $p(\text{win}) = (b / 80)^p$. Therefore the expected profit is:

$$\text{Expected Profit} = (80 - b) \times (b / 80)^p$$

As more people join the auction, the optimal bid goes up and the corresponding expected profit goes down. More specifically, as p increases, the optimal bid approaches $80. The reason is because the more people there are, the less likely you are to beat them, especially with a low bid. If you want to win an auction with lots of bidders, the best strategy is to bid the value of the item and not expect to profit.



Guiding Questions

1. For a given bid, b, what was the probability of winning a one-opponent auction? A two-opponent auction?
2. Based on this, what is the probability of winning an auction with p opponents?
3. How can we adjust our expected profit expression to work for *p* opponents?
4. When we graph $(80 - b) \times (b / 80)^p$ for larger and larger values of p, what happens to the max. expected profit?
5. What does this imply about the optimal bid for an auction with lots of people?

Checks for Understanding

- As the number of opponents goes up, the highest possible expected profit goes down. Why is this? *(As we know, the expected profit function is the product of the profit and the probability of winning. For an item whose value is $80, the profit is always 80 – b; even when more people join the auction, the profit associated with each bid doesn't change. Instead, what changes is the probability of winning; the more opponents you have in an auction, the less likely you are to win with any non-$80 bid. And when p(win) goes down, so does the expected profit.)*